**Lecture Sheet**

**On**

**Theory of Equation**

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**Polynomial:** A polynomial is an expression consisting of variables and coefficients that involves only the operations of addition, subtraction, multiplication and non-negative exponent of variables. A polynomial in variable *x* of the *n*-th degree is defined as,



where are independent of *x* and.

An equation consists of a polynomial is called a polynomial equation. A polynomial equation in variable *x* of the *n*-th degree is,

.

**Example:** 1). is a linear equation with single variable *x*.

2). is a quadratic equation with single variable *x*.

**Note:**

1. If , then  is called a polynomial of order (or degree) *n*.
2. If , then  is called monic polynomial of order *n*.
3. If , then  is called a polynomial of order zero that means a constant polynomial.
4. If , then is called a polynomial of order one (1) that means a linear polynomial.
5. If , then  is called a polynomial of order two means polynomial of degree 2 or Quadratic polynomial. The graph of a quadratic polynomial is a parabola.
6. If , then  is called a polynomial of order three means polynomial of degree 3 or Cubic polynomial.
7. If , then  is called a polynomial of order four means polynomial of degree 4 or by-quadratic polynomial.
8. If , then it is called zero polynomial with explicitly undefined degree. The graph of a zero polynomial is the *x*-axis.

Polynomials can be classified by the number of terms with nonzero coefficients such as a one-term polynomial is called a [monomial](https://en.wikipedia.org/wiki/Monomial); a two-term polynomial is called a [binomial](https://en.wikipedia.org/wiki/Binomial_(polynomial)); and a three-term polynomial is called a trinomial. The term "quadrinomial" is occasionally used for a four-term polynomial. A polynomial in one variable is called a [univariate](https://en.wikipedia.org/wiki/Univariate) polynomial; a polynomial in more than one variable is called a multivariate polynomial. A polynomial with two variables is called a bivariate polynomial.

**Remainder Theorem:** If  is a polynomial, then  is the remainder when is divided by .

This follows on substituting *h* for *x* in the identity,



where  and  are respectively the quotient and remainder in the division of  by and *R* is independent of *x*. If , then is a factor of .

**Roots of Equations:** Consider an equation of the type , where  is a polynomial. If  for , then  is called a root of .

The general equation of the *n*-th degree is written as,

.

Since it is an *n*-th degree equation so it has at least one root. This is the fundamental theorem of algebra.

**Theorem-01:** Proved that every equation of the *n*-th degree has exactly *n* roots.

**Proof:** Let .

By the fundamental theorem of algebra,  has at least one root. Let  be a root of . Then by the remainder theorem,  is divisible by ; we may therefore assume that,





where, .

Again let  be a root of ; as before,  is divisible by ; and we may assume that,



From (1) and (2), we get

.

Proceeding in this way, we can show that



where there are *n* linear factors on the right.

Hence, has *n* roots  and no others. **(Proved)**

**Imaginary Roots:** Let the coefficients of  be real. Then, if  is a root, so  is also a root. Therefore  is divisible by  that is, by .

Thus a polynomial in *x* with real coefficients can be resolved into factors which are linear or quadratic functions of *x* with real coefficients.

**Multiple Roots:** If where  is not divisible by , then  is called an *r*-multiple root of .

**Relation between the Roots and Coefficients of an Equation:** Consider a polynomial equation in *x* of the *n*-th degree,



Let  be the roots of equation (1), so



Equating the coefficients of the terms having same power, we get



.

These are the required relations between the roots and coefficients of the equation.

**Example:** If  are the roots of , then

, , and .

**Transformations of Equations:** Let be the roots of , and suppose that we require the equation whose roots are  where  is a given function of .

Let  and suppose that from this equation we can find x as a single-valued function of y, which we denote by . Transforming the equation by the substitution , we obtain , which is the required equation.

**Special cases:** The following transformations are often required. Let  be the roots of , then

1. the equation whose roots are  is .
2. the equation whose roots are  is .
3. the equation whose roots are  is .
4. the equation whose roots are  is .

**Problem-01:** If  are the roots of , then find the equation whose roots are

.

**Solution:** The given equation is,



The roots of equation (1) are .

Let 

.

From (1), we get



.

This is the required equation.

**Exercise:**

**Problem-01:** If  are the roots of , then find the equation whose roots are

1. .
2. .
3. .
4. .
5. .

**Problem-02:** If  are the roots of , then find the equation whose roots are

1. .
2. .

**Theorem-02:** State and prove Descarte’s Rule of Signs.

**Statement:** The equation  cannot have more positive roots than  has changes of sign, or more negative roots than  has changes of sign.

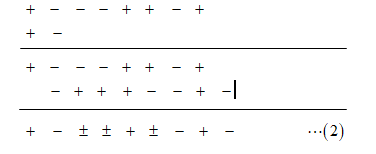
**Proof:** Let  be a polynomial equation and no term is missing in the polynomial . Also let the signs of the different terms of this equation are,



Again let 

To prove the first part, we shall show that  has at least one more change of sign than .

The signs of the terms of the linear polynomial are “”. To obtain the signs of the polynomial , multiplying the signs of the polynomial  by the sings of the linear polynomial . Which are given as,



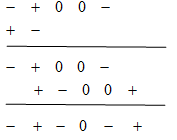
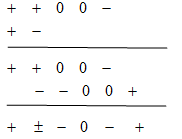
where  indicates that the sign may be  or , or that the corresponding term is zero.

In the diagram of corresponding signs, observe that

1. If the *r* th sign of  is a continuation, the *r* th sign of  is ambiguous.
2. Unlike signs precede and follow a single ambiguity or a group of ambiguities.
3. A change of sign is introduced at the end of .

On account of (i) and (ii) ,  has at least as many changes of sign as , even in the most unfavorable case in which all the ambiguities are continuations, and on account of (iii)  has certainly one more change of sign than .

That no changes of sign are lost on account of any terms which may be missing from  appears on considering such instances as,

Thus,  has at least one more change of sign than .

Next let  where  are the positive roots of . If  is multiplied in succession by , each multiplication introduces at least one change of sign. Hence  has at least as many changes of sign as has positive roots.

Again, the negative roots of  are the positive roots of , with their signs changed. Hence the second part of the theorem follows from the first. **(Proved)**

**Note:** The equation  has 2 continuations, and 4 changes of sign, the continuations occurring at the terms , and the changes at .

**Problem-02:** If 1 and 7 are two roots of , then solve the equation..

**Solution:** The given equation is,



Let the other two roots of equation (1) are . From the relation of coefficients and roots, we have





And 



We know, 





Solving (2) and (4), we get

.

The roots or solution of the given equation are 1, 3, 5, 7.

**Problem-03:** Solve  where the roots are in arithmetic progression.

**Solution:** The given equation is,



Since the roots are in arithmetic progression so the roots of equation (1) are . From the relation of coefficients and roots, we have



And 









Now using the value of and any one value of , we have

.

The roots or solution of the given equation are  .

**Problem-04:** Solve  where the roots are in arithmetic progression.

**Solution:** The given equation is,



Since the roots are in arithmetic progression so the roots of equation (1) are . From the relation of coefficients and roots, we have



And 



















Now using the value of and any one value of , we have



The roots or solution of the given equation are  .

**Problem-05:** Solve  where the roots are in geometric progression.

**Solution:** The given equation is,



Since the roots are in geometric progression so the roots of equation (1) are . From the relation of coefficients and roots, we have



And 



Using the value of  in (2), we get









Now using the value of and any one value of , we have

.

The roots or solution of the given equation are  .

**Problem-06:** Solve  where the roots are in geometric progression.

**Solution:** The given equation is,



Since the roots are in geometric progression so the roots of equation (1) are . From the relation of coefficients and roots, we have



and 

From (3), we get

.

From (2), we get

















.

Now 



Using the value of , we have

.

The roots or solution of the given equation are  .

**Problem-07:** Solve , where  is a root of this equation.

**Solution:** The given equation is,



Since  is a root of the equation (1) so  is also root of this equation. Let the other two roots of this equation are . From the relation of coefficients and roots, we have





and 



We know, 





Solving (2) and (4), we get

.

The roots or solution of the given equation are .

**Problem-08:** Solve , where  is a root of this equation.

**Solution:** The given equation is,



Since  is a root of the equation (1) so  is also root of this equation. Let the other two roots of this equation are . From the relation of coefficients and roots, we have





and 



We know, 





Solving (2) and (4), we get

.

The roots or solution of the given equation are .

**Exercise:**

**Problem-03:** Solve  where the roots are in arithmetic progression.

**Problem-04:** Solve  where the roots are in geometric progression.

**Problem-05:** Solve , where  is a root of this equation.

**Problem-06:** Solve , where  is a root of this equation.